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Chaotic dynamics for perturbations of infinite-dimensional Hamiltonian systems☆

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1. Introduction

This paper deals with chaotic behaviour for perturbations of infinite-dimensional autonomous Hamiltonian systems modelling a compressed forced beam and a Sine–Gordon equation.

First we consider a PDE of the type

$$(\mathscr{H}_\varepsilon) \quad w_{tt} + w_{zzzz} + \gamma w_{zz} - \kappa \left(\int_0^1 w_z^2(t,\xi)\,\mathrm{d}\xi \right) w_{zz} = \varepsilon(P(t,w) - \delta w_t), \quad \varepsilon \geq 0,$$

where $w(t,z) \in \mathbb{R}$ is the transverse deflection of the axis of the beam; $w(t,0) = w(t,1) = w_{zz}(t,0) = w_{zz}(t,1) = 0$, γ is an external load, $\kappa > 0$ is a ratio indicating the extensional rigidity and δ is the damping.

The first result on the existence of a chaotic dynamics for system $(\mathscr{H}_\varepsilon)$ has been given by Holmes and Marsden in [10] for a specific periodic forcing perturbation of the type $P(t, w(t,z)) = f(z)\cos(\omega t)$. They use the theory of invariant manifolds and of non-linear semigroups in order to extend the classical Melnikov approach for *planar* ordinary differential equations to system $(\mathscr{H}_\varepsilon)$.

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More precisely, the function $w(t,z) = 0$ is an equilibrium solution of the unperturbed Hamiltonian system $(\mathscr{H}_0)$ with Liapounov exponents (i.e. the eigenvalues of the linearized system at the equilibrium) given by

$$\lambda_j^2 = j^2\pi^2(\gamma - j^2\pi^2) \quad j = 1,2,\ldots, \tag{1.1}$$

and eigenvectors given by $\sin(j\pi z)$ (called the fundamental modes). Holmes and Marsden assume that

$(\gamma 1)$ $\quad \pi^2 < \gamma < 4\pi^2,$

and then the equilibrium $w = 0$ is unstable with one positive λ_1 and one negative $-\lambda_1$ eigenvalue, and possesses an infinite-dimensional center manifold. Using the Fourier expansion $w(t,z) = \sum_{j\geq 1} w_j(t)\sin(j\pi z)$ w.r.t. the basis of the eigenvectors $\sin(j\pi z)$, $(\mathscr{H}_0)$ is equivalent to an infinite sequence of second-order Hamiltonian differential equations, one for each modal coefficient $w_j(t)$, given by

$$\begin{aligned}
&-\ddot{w}_1 + \lambda_1^2 w_1 = (\kappa\pi^4/2)w_1^3 + N_1(w),\\
&\ldots,\\
&\ddot{w}_j + |\lambda_j^2| w_j = N_j(w) \quad \text{for } j \geq 2,\\
&\ldots
\end{aligned}$$

where $N_1(w),\ldots,N_j(w),\ldots$ $j\geq 2$ are non-linear coupling terms of third order (see system (2.2)) for a more precise expression). Under hypothesis $(\gamma 1)$ the equation for w_1 is of Duffing's type while the equations for w_j $(j\geq 2)$ behave near the equilibrium like harmonic oscillators, with frequencies $|\lambda_j|$.

The one-dimensional unperturbed stable and unstable manifolds $W_0^{s,u}$, living on the first mode $\sin(\pi z)$, coincide and support the family of homoclinic solutions $u_\theta(t,z) = x_\theta(t)\sin(\pi z)$, where $x_\theta(t) = x_0(t-\theta)$ and $x_0(t) = \sqrt{4\lambda_1^2/\kappa\pi^4}\,\mathrm{sech}(\lambda_1 t)$ is the homoclinic solution of Duffing's equation $-\ddot{x} + \lambda_1^2 x = (\kappa\pi^4/2)x^3$ with $\dot{x}_0(0) = 0$, $x_0(t) > 0$.

Assume the non-resonance condition $\omega^2 \neq |\lambda_j^2|$ (for $j = 2,3,\ldots$) between the forcing frequency ω and the frequencies $|\lambda_j|$ of the small oscillations of the beam near the equilibrium. Then, when ε is small enough, from the state w=0 branches a periodic orbit $\gamma_\varepsilon = O(\varepsilon)$, with stable and unstable manifolds $W_\varepsilon^{u,s}$ such that $codim(W_\varepsilon^s) = dim(W_\varepsilon^u) = 1$. Since the perturbation term $f(z)\cos(\omega t)$ is very simple, Holmes and Marsden are able to compute the Melnikov function of the system explicitly. This allows to verify that, for $\varepsilon\delta \neq 0$ small enough, the Melnikov function has simple zeroes (a property difficult to be checked for a general perturbation terms P) and then that W_ε^s and W_ε^u intersect transversally. By an application of the Smale–Birkhoff theorem, the existence of horseshoes in the system follows.

The same techniques of [10] have been applied by Holmes in [9] to a Sine–Gordon equation like

$(\mathscr{SG}_\varepsilon)$ $\quad \psi_{tt} - \psi_{zz} + \sin\psi = \varepsilon(P(t,\psi) - \delta\psi_t) \quad \varepsilon \geq 0,$

with $\psi_z(t,0) = \psi_z(t,1) = 0$. In this case, the unperturbed system possesses two equilibrium solutions $\psi^{\pm}(t) = \pm\pi$. The Liapounov exponents of these equilibria are $\lambda_j^2 = 1 - j^2\pi^2$ with eigenvectors $\cos(j\pi z)$ for $j = 0,1,\ldots$. Then also in this case the stationary states are unstable with one positive and one negative eigenvalue, possess an infinite dimensional center manifold, and are connected by two families of heteroclinic orbits $x_\theta^{\pm}(t)$ (the separatrices of the standard pendulum). For specific T-periodic forcing perturbations $P(\cdot,\psi)$ the equilibrium states $\pm\pi$ perturb to small periodic orbits $\gamma_\varepsilon^{\pm}$ and one can explicitly compute the corresponding Melnikov functions. By inspection, one can infer that they possess simple zeroes and then that there exist transversal heteroclinic orbits for $(\mathscr{S}\mathscr{G}_\varepsilon)$ which imply a chaotic dynamics.

The role of the damping term $\varepsilon\delta w_t$ is the following. For $\varepsilon\delta \neq 0$, the "center directions" $\sin(j\pi z)$, $j\geq 2$, for $(\mathscr{H}_\varepsilon)$ (resp. $\cos(j\pi z)$, $j\geq 1$ for $(\mathscr{S}\mathscr{G}_\varepsilon)$) become foci, i.e. $w=0$ (resp. $\psi=\pm\pi$) becomes an hyperbolic equilibrium whose Liapounov exponents have real part $-(\varepsilon\delta/2)$ for $j\geq 2$ (resp. for $j\geq 1$). The undamped case ($\delta=0$) would require an infinite-dimensional version of Arnold diffusion (see [4]).

In recent years, starting with [1], another functional approach to study existence and multiplicity of homoclinic orbits to a hyperbolic equilibrium for perturbed Lagrangian and Hamiltonian systems in $\mathbb{R}^n$ has been developed (see also [3,8]). Homoclinic solutions are found as critical points of the action functional $f_\varepsilon = f_0 + \varepsilon f_1$. Let us assume that the unperturbed functional f_0 possesses a finite-dimensional manifold Z of critical points (unperturbed homoclinic solutions) satisfying the non-degeneracy condition $ker\, f_0''(z) = T_z Z\ \forall z\in Z$; through a Liapounov–Schmidt-type reduction, the search of critical points for the action functional f_ε is reduced to look for critical points of f_ε restricted to a finite-dimensional manifold Z_ε near Z. It turns out that, up to a constant, the restriction of f_ε to Z_ε is very close to the Poincaré–Melnikov function (the primitive of the Melnikov function) and then a critical point of this latter function gives rise to a critical point of the action functional f_ε, and hence to a homoclinic solution. In [7], the approach of [1] has been generalized: when the Poincaré–Melnikov function is oscillating, they find homoclinic orbits of multibump type implying a chaotic dynamics in the system (in particular the topological entropy is positive, see [12]). In [5], the results of [7] have been extended proving the existence of infinitely many heteroclinic orbits for perturbed Lagrangean systems possessing two or more hyperbolic equilibrium states.

The aim of this paper is to extend the results of Holmes and Marsden [10] and of Holmes [9]; more precisely the improvements of our approach are the followings:

(1) We do not require any restriction on the time dependence of the perturbation $P(\cdot,w)$, such as periodicity, almost periodicity, etc.
(2) In order to obtain homoclinics for $(\mathscr{H}_\varepsilon)$, resp. heteroclinics for $(\mathscr{S}\mathscr{G}_\varepsilon)$, the Melnikov function can possess zeroes that are just "*topologically simple*" (see Definition 4.1).

In this case, the corresponding homoclinics will not be transversal and one cannot invoke (even in the periodic case) the Smale–Birkhoff theorem in order to prove the existence of chaotic trajectories of multibump type.

We note that our condition, called "Melnikov oscillating", needed to find a chaotic behaviour, is always satisfied when $P(\cdot,w)$ is periodic, quasi-periodic or almost-periodic in time and when the Melnikov function is non-constant.

(3-i) We can apply our method when $\gamma\in(m^2\pi^2,(m+1)^2\pi^2)$ for $m=2,3,\dots$. In this case, the first m equations for the modal coefficients w_j become a system of m-coupled Duffing equations (the other directions are still centers), $dim\, W_0^{\mathrm{s}} = dim\, W_0^{\mathrm{u}} = m$ and there exists at least one homoclinic solution of $(\mathscr{H}_0)$ which satisfies a suitable non-degeneracy (transversality) condition. For ε small enough we prove the existence of multibump homoclinics for the perturbed system which bifurcate from this unperturbed homoclinic orbit.

(3-ii) We can also apply our approach when w is vector valued.

The techniques of Holmes and Marsden cannot be applied in cases (3-i) and (3-ii) since, adapting the classical Melnikov approach, they work only for perturbations of planar systems.

In order to prove our results we cannot apply directly the method developed in [1,7] (see also [2,6]). Indeed, apart that ($\mathscr{H}_\varepsilon$) (resp. ($\mathscr{S}\mathscr{G}_\varepsilon$)) is infinite dimensional and the equation is not variational, the main difference between system ($\mathscr{H}_\varepsilon$) (resp. ($\mathscr{S}\mathscr{G}_\varepsilon$)) and the ones considered in [1,7] (resp. [5]) is that, $w=0$ (resp. $\psi=\pm\pi$) is not a hyperbolic point of equilibrium for ($\mathscr{H}_0$) (resp. $\mathscr{S}\mathscr{G}_0$). This requires modifications in the proofs: for $\varepsilon\delta\neq 0$ the damping term produces an $\varepsilon\delta$-hyperbolicity in the equilibrium but then the unperturbed homoclinics (resp. heteroclinics) u_θ (resp. $x_\theta^{\pm}$) are just ε-pseudo solutions of ($\mathscr{H}_\varepsilon$) (resp. $\mathscr{S}\mathscr{G}_\varepsilon$) and system ($\mathscr{H}_\varepsilon$) (resp. $\mathscr{S}\mathscr{G}_\varepsilon$) is no longer Hamiltonian. However, using the contraction mapping theorem, it is still possible to perform, for ε small enough, a finite-dimensional reduction near the unperturbed solutions u_θ (resp. $x_\theta^{\pm}$) analogue to the one of [1,7] (resp. [5]).

After this paper was completed, we learned about a paper by McLaughlin and Shatah [11] which deals with the persistence of homoclinics for the perturbed Sine–Gordon equation. The methods that the authors use are similar to ours. They consider as unperturbed homoclinic the "breather" solution while we consider an unperturbed homoclinic which depends only on the time variable. It is easy to see that our method applies also starting from an unperturbed breather homoclinic. Unlike the paper [11], we consider also the existence of infinitely many homoclinics and of solutions with infinitely many bumps which imply the existence of a chaotic dynamics. We also learned about a forthcoming paper by Shatah and Zeng [13], where the McLaughlin and Shatah result has been proved to hold, still for 1-bump solutions, for more general perturbation terms (still periodic in time).

2. Functional setting

We shall use as "phase space" for the evolution equation ($\mathscr{H}_\varepsilon$) one of the following Banach spaces of functions of the spatial variable $z\in[0,1]=I$ defined, for any integer $k\in\mathbb{N}$ by

$$C_D^k(I)=\left\{u(z)=\sum_{j\geq 1}u_j\sin(j\pi z)\,\middle|\,\sum_{j\geq 1}|u_j|j^k<+\infty\right\}\quad\text{with norm } \|u\|_{C_D^k}=\sum_{j\geq 1}|u_j|j^k\,.$$

We clearly have that $C_D^{k'}(I)\subset C_D^k(I)$ if $k<k'$.

We look for solutions homoclinic to 0 of $(\mathscr{H}_\varepsilon)$, i.e. solutions with $\|w(t)\|_{C^4_D}$, $\|\dot{w}(t)\|_{C^4_D} \to 0$ as $|t| \to +\infty$. Then we define the following spaces of curves in the phase space:

$$E_k = \left\{ w(t,z) = \sum_{j \geq 1} w_j(t)\sin(j\pi z) \,|\, w_j(\cdot) \in C_0(\mathbb{R},\mathbb{R}) \text{ (i.e. } w_j(t) \to 0 \text{ as } t \to \pm\infty) \right.$$

$$\left. \text{and } \sum_{j \geq 1} \|w_j\|_\infty j^k < +\infty \right\},$$

with norm $\|w\|_{E_k} = \sum_{j \geq 1} \|w_j\|_\infty j^k$, and we set

$$\tilde{E}_k = \left\{ u \in E_k \,\middle|\, \int_0^1 u(t,z)\sin(\pi z)\,\mathrm{d}z = 0 \right\} \quad \text{so that } E_k = C_0(\mathbb{R}) \oplus \tilde{E}_k\,.$$

We will also define

$$B_k = \left\{ w(t,z) = \sum_{j \geq 1} w_j(t)\sin(j\pi z) \,|\, w_j(\cdot) \in BC(\mathbb{R},\mathbb{R}) \text{ (bounded continuous)} \right.$$

$$\left. \text{and } \sum_{j \geq 1} \|w_j\|_\infty j^k < +\infty \right\}$$

with norm $\|w\|_{B_k} = \sum_{j \geq 1} \|w_j\|_\infty j^k$; clearly, $E_k \subset B_k$ and $E_k \subset C_0(\mathbb{R}, C^k_D(I)) = \{w : \mathbb{R} \to C^k_D(I) \,|\, \text{continuous with } \|w(t)\| \to 0 \text{ as } t \to \pm\infty\}$. All solutions homoclinic to $w = 0$ are contained in E_4.

We assume

(P1) $P(t,w) \in C^1(\mathbb{R} \times C^4_D, C^2_D)$ with $P(t,0)=0$, $D_w P(t,0)=0 \in \mathscr{L}(C^4_D(I), C^2_D(I))$, $P(\cdot,w)$, $D_w P(\cdot,w) \in L^\infty(\mathbb{R})$ on bounded sets of C^4_D and such that there is a $\rho_0 > 0$ such that in $B(0,\rho_0) = \{w \in C^4_D(I) \,|\, \|w\|_{C^4_D(I)} \leq \rho_0\}$ $D_w P(t,w)$ is Λ-Lipschitz continuous, i.e. for all $w, \bar{w} \in B(0,\rho_0)$

$$\|D_w P(t,w) - D_w P(t,\bar{w})\|_{\mathscr{L}(C^4_D, C^2_D)} \leq \Lambda \|w - \bar{w}\|_{C^4_D}.$$

It will be useful to write the perturbation P as $P(t,w) = \sum_{j \geq 1} p_j(t,w)\sin(j\pi z) = p_1(t,w)\sin(\pi z) + P_2(t,w)$, where $P_2(t,w) = \sum_{j \geq 2} p_j(t,w)\sin(j\pi z) \in \tilde{E}_2$.

In this section, we assume hypothesis $(\gamma 1)$.

By (P1), $w = 0$ is as equilibrium solution of $(\mathscr{H}_\varepsilon)$ and the linearized equation at the equilibrium is

$$(\mathscr{L}_\varepsilon) \quad w_{tt} + \varepsilon\delta w_t + w_{zzzz} + \gamma w_{zz} = 0.$$

with $w(t,0) = w(t,1) = w_{zz}(t,0) = w_{zz}(t,1) = 0$. Setting $w(t,z) = w_0(z)\mathrm{e}^{\lambda t}$ and solving for the eigenvalues and eigenvectors we obtain

$$\lambda^2 w_0(z) + \lambda\varepsilon\delta w_0(z) + w_0^{(\mathrm{iv})}(z) + \gamma w_0''(z) = 0$$

with $w_0(0)=w_0(1)=w_0''(0)=w_0''(1)=0$. Hence that $\sin(j\pi z)$ are the eigenvectors, and the eigenvalues are the solutions of $\lambda^2+\varepsilon\delta\lambda-\lambda_j^2=0$, that is

$$\lambda_{\varepsilon,j}^{\pm}=\tfrac{1}{2}\left[-\varepsilon\delta\pm\sqrt{\varepsilon^2\delta^2+4\lambda_j^2}\right]\quad j=1,2,\ldots, \tag{2.1}$$

where λ_j^2 are the eigenvalues of $(\mathscr{L}_0)$, given by (1.1). Clearly expanding a solution of $(\mathscr{L}_\varepsilon)$ w.r.t. the basis of the eigenfunctions $\sin(j\pi z)$, i.e. setting $w(t,z)=\sum_{j\geq 1}w_j(t)\sin(j\pi z)$, we obtain an infinite number of *decoupled* second-order equations one for each modal coefficient $w_j(t)$ given by

$$-\ddot{w}_j-\varepsilon\delta\dot{w}_j+\lambda_j^2w_j=0,\quad j\geq 1.$$

It is useful to study system $(\mathscr{H}_\varepsilon)$ separatly along the hyperbolic mode $\sin(\pi z)$ and the ε-hyperbolic modes $\sin(j\pi z)$ for $j\geq 2$ along which the dynamics is quite different. We write

$$w(t,z)=x(t)\sin(\pi z)+u(t,z)$$

with $u(t,z)=\sum_{j\geq 2}u_j(t)\sin(j\pi z)\in\tilde{E}_4$ and substitute in $(\mathscr{H}_\varepsilon)$. We obtain the following system in the variables (x,u):

$$(\mathscr{S}_\varepsilon)\quad\begin{cases}-\ddot{x}-\varepsilon\delta\dot{x}+\beta x=\alpha x^3+\kappa\pi^2x\left(\displaystyle\int_0^1u_z^2(t,\xi)\,\mathrm{d}\xi\right)-\varepsilon p_1(t,x,u),\\[2ex] u_{tt}+\varepsilon\delta u_t+u_{zzzz}+\left(\gamma-\frac{\kappa x^2\pi^2}{2}\right)u_{zz}=\kappa\left(\displaystyle\int_0^1u_z^2(t,\xi)\,\mathrm{d}\xi\right)u_{zz}+\varepsilon P_2(t,x,u),\end{cases}$$

where $\beta=\lambda_1^2$, $\alpha=\kappa\pi^4/2$ and with a small abuse of notation, we have set $p_1(t,x,u)=p_1(t,x\sin(\pi z)+u)$ and $P_2(t,x,u)=P_2(t,x\sin(\pi z)+u)$. In "coordinates", system $(\mathscr{S}_\varepsilon)$ has the form

$$\begin{aligned}&-\ddot{x}-\varepsilon\delta\dot{x}+\beta x=\alpha x^3+\kappa\pi^2x\left(\int_0^1u_z^2(t,\xi)\,\mathrm{d}\xi\right)-\varepsilon p_1(t,x,u),\\&\ldots\\&\ddot{u}_j+\varepsilon\delta\dot{u}_j+\left[j^2\pi^2(j^2\pi^2-\gamma)+j^2\frac{\kappa\pi^4}{2}x^2(t)\right]u_j\\&\quad=-\kappa(j\pi)^2u_j\left(\int_0^1u_z^2(t,\xi)\,\mathrm{d}\xi\right)+\varepsilon p_j(t,x,u)\quad\text{for } j\geq 2.\\&\ldots\end{aligned}\tag{2.2}$$

An homoclinic for system $(\mathscr{S}_\varepsilon)$ is a solution with $x(t)$, $\dot{x}(t)\to 0$ and $\|u(t)\|_{C_D^4}$, $\|\dot{u}(t)\|_{C_D^4}\to 0$ as $|t|\to+\infty$.

In order to apply the contraction mapping theorem, we consider the linear Green operators L_ε and G_ε which are, respectively, the inverses of the differential operators

$$\partial_{tt}+\varepsilon\delta\partial_t+\partial_{zzzz}+\left(\gamma-\frac{\kappa x_\theta^2(t)\pi^2}{2}\right)\partial_{zz}\quad\text{and}\quad-\frac{\mathrm{d}^2}{\mathrm{d}t^2}-\varepsilon\delta\frac{\mathrm{d}}{\mathrm{d}t}+\beta$$

with zero Dirichlet boundary conditions at $t\to\pm\infty$, which allow us to write system $(\mathscr{S}_\varepsilon)$ in the form of an integral equation. The following lemmas can be proved.

Lemma 2.1. *There exist positive constants C_1, C_2 such that for all $f \in \tilde{E}_2$ there exists a unique solution $u \in \tilde{E}_4$ of*

$$u_{tt} + \varepsilon\delta u_t + u_{zzzz} + \left(\gamma - \frac{\kappa x_\theta^2(t)\pi^2}{2}\right) u_{zz} = f \tag{2.3}$$

given by $L_\varepsilon(f) := u = \sum_{j \geq 2} u_j(t)\sin(j\pi z)$ with

$$u_j(t) = -\left(\int_{-\infty}^{t} \mathrm{e}^{(\varepsilon\delta/2)(s-t)} w_j(s) f_j(s)\,\mathrm{d}s\right) v_j(t)$$

$$+ \left(\int_{-\infty}^{t} \mathrm{e}^{(\varepsilon\delta/2)(s-t)} v_j(s) f_j(s)\,\mathrm{d}s\right) w_j(t), \tag{2.4}$$

where $\|w_j\|_\infty \leq C_1/j^2$ and $\|v_j\|_\infty \leq C_2$. There exists $C_3 > 0$ such that $L_\varepsilon : \tilde{E}_2 \to \tilde{E}_4$ satisfies

$$\|L_\varepsilon f\|_{E_4} \leq \frac{C_3}{\varepsilon\delta}\|f\|_{E_2}.$$

Moreover, this estimate can be improved for exponentially decaying functions: if ϕ is a real function satisfying $|\phi(t)| \leq a\mathrm{e}^{-b|t|}$ for some $a, b > 0$ and $f \in E_2$ then

$$\|L_\varepsilon(f\phi)\|_{E_4} \leq \frac{a}{b} C_3' \|f\|_{E_2}$$

for a suitable constant C_3' which does not *depend on ε.*

Lemma 2.2. *Let $f \in C_0(\mathbb{R})$. There exists a unique C^2-solution $u \in C_0(\mathbb{R})$*

$$-\ddot{u} - \varepsilon\delta\dot{u} + \beta u = f$$

given by

$$u := G_\varepsilon(f) = \frac{1}{\sqrt{(\varepsilon\delta)^2 + 4\beta}}\left[\int_t^{+\infty} f(s)\mathrm{e}^{\lambda_{\varepsilon,1}^+(t-s)}\,\mathrm{d}s + \int_{-\infty}^{t} f(s)\mathrm{e}^{\lambda_{\varepsilon,1}^-(t-s)}\,\mathrm{d}s\right],$$

where $\lambda_{\varepsilon,1}^{\pm} = (\frac{1}{2})(-\varepsilon\delta \pm \sqrt{\varepsilon^2\delta^2 + 4\beta})$ are the roots of $p(\lambda) = -\lambda^2 - \varepsilon\delta\lambda + \beta$. There exist $C_4, C_5 > 0$ such that
(i) $\|G_\varepsilon - G_0\|_{\mathscr{L}(C_0,C_0)} \leq C_4\varepsilon$ *as* $\varepsilon \to 0$,
(ii) $\|G_\varepsilon\|_{\mathscr{L}(C_0,C_0)} \leq C_5$ *as* $\varepsilon \to 0$.

Finally, we consider the non-linear operator $S_\varepsilon(x,u) : C_0(\mathbb{R}) \times \tilde{E}_4 \to C_0(\mathbb{R}) \times \tilde{E}_4$ given by

$$S_\varepsilon(x,u) = \begin{pmatrix} x - G_\varepsilon\left(\alpha x^3 + \kappa\pi^2 x\left(\int_0^1 u_z^2\right) - \varepsilon p_1(t,x,u)\right) \\ u - L_\varepsilon\left(\frac{\kappa\pi^2}{2}(x^2 - x_\theta^2)u_{zz} + \kappa\left(\int_0^1 u_z^2\right)u_{zz} + \varepsilon P_2(t,x,u)\right) \end{pmatrix}.$$

It can be easily seen that a non-trivial zero of $S_\varepsilon(x,u)$ in $C_0(\mathbb{R}) \times \tilde{E}_4$ is an homoclinic solution to 0 of system ($\mathscr{S}_\varepsilon$), i.e. a solution with satisfying also $\|\dot{x}(t)\|_\infty, \|\dot{u}(t)\|_{C_D^4(I)} \to 0$ as $|t| \to +\infty$.

3. The finite-dimensional reduction

The unperturbed autonomous system $(\mathscr{H}_0)$ possesses a one-dimensional manifold of homoclinic solutions given by $x_\theta \sin(\pi z)$. Equivalently,

$$Z = \{(x_\theta(\cdot), 0) = (x_0(\cdot - \theta), 0) \mid \theta \in \mathbb{R}\} \subset C_0(\mathbb{R}) \times \tilde{E}_4$$

is a one-dimensional manifold of homoclinic solutions for system $(\mathscr{S}_0)$. Its tangent space at $(x_\theta, 0)$ is given by $T_{(x_\theta,0)}Z = span(\dot{x}_\theta, 0)$.

Remark 3.1. The non-degeneracy condition $span(\dot{x}_\theta) = \mathscr{K}$ holds, where $\mathscr{K}$ is the linear space of the solutions v of the linear equation $-\ddot{v} + \beta v = 3\alpha x_\theta^2 v$ with $v(t) \to 0$ as $|t| \to +\infty$. $\dot{x}_\theta \in \mathscr{K}$ and $dim\, \mathscr{K} = dim\, T_p W_0^{\rm s} \cap T_p W_0^{\rm u}$, for any $p \in \xi$, where ξ, denotes the homoclinic trajectory $(x_0, \dot{x}_0)$ in the phase space $\mathbb{R}^2$ (see [2]). Since $W_0^{\rm s} = W_0^{\rm u}$ and $dim\, W_0^{\rm u,s} = 1$, we conclude that $span(\dot{x}_\theta) = \mathscr{K}$.

In order to study the dynamics of $(\mathscr{S}_\varepsilon)$ in a neighborhood of $(x_\theta, 0)$ we perform, for ε small enough, a Liapounov–Schmidt-type finite-dimensional reduction using the contraction mapping theorem. This is the main lemma. We will always consider $\varepsilon \in (0, 1)$ in what follows. The following lemma holds.

Lemma 3.1. *There are* $\varepsilon_1, C_6 > 0$ *and smooth functions* $(w(\varepsilon, \theta), y(\varepsilon, \theta), \mu(\varepsilon, \theta))$: $(-\varepsilon_1, \varepsilon_1) \times \mathbb{R} \to C_0(\mathbb{R}, \mathbb{R}) \times \tilde{E}_4 \times \mathbb{R}$ *such that*:
(i) $S_\varepsilon(x_\theta + w_\varepsilon(\theta), y_\varepsilon(\theta)) = (\mu_\varepsilon(\theta) G_\varepsilon(\dot{x}_\theta), 0)$;
(ii) $(w_\varepsilon(\theta), \dot{x}_\theta)_{L^2} = 0$;
(iii) $\|w_\varepsilon(\theta)\|_\infty, \|y_\varepsilon(\theta)\|_{E_4} \le C_6 \varepsilon$ *for all* $0 < \varepsilon \le \varepsilon_1$ *and* $\theta \in \mathbb{R}$.

Proof. Let us define the function

$$H : \mathbb{R} \times \mathbb{R} \times C_0(\mathbb{R}) \times \mathbb{R} \times \tilde{E}_4 \to C_0(\mathbb{R}) \times \mathbb{R} \times \tilde{E}_4$$

with components $H_1(\varepsilon, \theta, w, \mu, y) \in C_0(\mathbb{R}) \times \mathbb{R}$ and $H_2(\varepsilon, \theta, w, \mu, y) \in \tilde{E}_4$ given by

$$H_1 = \left(w - \alpha(G_\varepsilon(x_\theta + w)^3 - G_0(x_\theta^3)) - G_\varepsilon \left(\kappa\pi^2 (x_\theta + w) \left(\int_0^1 y_z^2 \right) \right.\right.$$
$$\left.\left. -\varepsilon\ p_1(t, x_\theta + w, y) \right) - \mu G_\varepsilon(\dot{x}_\theta), (w, \dot{x}_\theta)_{L^2} \right),$$

$$H_2 = y - L_\varepsilon \left(\frac{\kappa\pi^2}{2} (2x_\theta w + w^2) y_{zz} + \kappa \left(\int_0^1 y_z^2 \right) y_{zz} + \varepsilon P_2(t, x_\theta + w, y) \right).$$

In order to satisfy conditions (i) and (ii), we must find (w, μ, y) such that

$$H(\varepsilon, \theta, w, \mu, y) = 0 \tag{3.1}$$

(note that we cannot put $\varepsilon = 0$ since the operator L_ε would not be defined any more).

Let B_ρ be the ball in $C_0(\mathbb{R}) \times \mathbb{R} \times \tilde{E}_4$ with norm $|||(w, \mu, y)||| = \max(\|w\|_\infty, |\mu|, \|y\|_{E_4})$, of centre 0 and radius ρ that is $B_\rho = \{(w, \mu, y) \mid |||(w, \mu, y)||| \le \rho\}$. We will solve Eq. (3.1) by means of the contraction-mapping theorem, proving that, provided ε and ρ are small enough, there is a unique $(w(\varepsilon,\theta),\mu(\varepsilon,\theta),y(\varepsilon,\theta))\in B_\rho$ such that $H(\varepsilon,\theta,w(\varepsilon,\theta),\mu(\varepsilon,\theta),y(\varepsilon,\theta))=0$. We will assume $0<\rho\le 1$.

In order to put (3.1) as a fixed-point problem we consider

$$\frac{\partial H_1}{\partial(w,\mu)}_{|(\varepsilon,\theta,0,0,0)}[w,\mu]=(w-G_\varepsilon(3\alpha x_\theta^2 w)+\varepsilon G_\varepsilon(\partial_x p_1(t,x_\theta,0)w) - \mu G_\varepsilon(\dot x_\theta),(w,\dot x_\theta)_{L^2}) \tag{3.2}$$

which is "Id + Compact". We shall use the following abbreviation $b(\varepsilon,\theta)=\partial H/\partial(w,\mu)_{|(\varepsilon,\theta,0,0,0)}\in\mathscr{L}(C_0(\mathbb{R})\times\mathbb{R})$.

The unique homoclinic solution v of the linearized system

$$-\ddot v+\beta v=\alpha 3x_\theta^2(t)v-\mu\dot x_\theta,$$
$$(v,\dot x_\theta)_{L^2}=0,$$

which tends to 0 as $t\to\pm\infty$, is 0. Indeed, multiplyng by $\dot x_\theta$ the first equation and integrating on $\mathbb{R}$ we obtain

$$\int_{\mathbb{R}}(-\ddot v+\beta v-\alpha 3x_\theta^2(t)v)\dot x_\theta\,\mathrm{d}t=\mu\int_{\mathbb{R}}\dot x_0^2(t)\,\mathrm{d}t.$$

Integrating by parts the first member we see that it is 0 and therefore $\mu=0$. Then, by Remark 3.1 $v=c\dot x_\theta$ and, since $(v,\dot x_\theta)_{L^2}=0$, we get $c=0$.

It follows that $b(0,\theta)$ is injective and hence invertible. Easy estimates using Lemma 2.2 show that

$$\exists\bar C,\bar\varepsilon>0\quad\text{such that } \|b^{-1}(\varepsilon,\theta)\|\le\bar C\quad\forall 0<\varepsilon\le\bar\varepsilon. \tag{3.3}$$

$H(\varepsilon,\theta,w,\mu,y)=0$ is equivalent to $F(w,\mu,y)=(w,\mu,y)$ with

$$F(w,\mu,y)=(-b^{-1}(\varepsilon,\theta)H_1(\varepsilon,\theta,0,0,0)-b^{-1}(\varepsilon,\theta)R(\varepsilon,\theta,w,\mu,y),$$
$$\times L_\varepsilon(N(t,w,y)+\varepsilon P_2(t,x_\theta+w,y)))\quad\text{with}$$

$$R=H_1(\varepsilon,\theta,w,\mu,y)-H_1(\varepsilon,\theta,0,0,0)-b(\varepsilon,\theta)[w,\mu]$$
$$=(G_\varepsilon(-\alpha(3x_\theta w^2+w^3)+\kappa\pi^2(x_\theta+w)\left(\int_0^1 y_z^2\right)-\varepsilon(p_1(t,x_\theta+w,y)$$
$$-p_1(t,x_\theta,0)-\partial_x p_1(t,x_\theta,0)w),0)$$

and

$$N(t,w,y)=\frac{\kappa\pi^2}{2}(2x_\theta w+w^2)y_{zz}+\kappa\left(\int_0^1 y_z^2\right)y_{zz}.$$

We will find $\varepsilon_1>0$ and $C_6>0$ such that, if $0<\varepsilon\le\varepsilon_1$ and if $\rho=C_6\varepsilon$, then
(i) $\overline{F(B_\rho)}\subset B_\rho$;
(ii) F is a contraction on B_ρ.

First of all we have, using Lemmas 2.1, 2.2 and (P1) that

$$H(\varepsilon,\theta,0,0,0) = (-\alpha(G_\varepsilon(x_\theta^3) - G_0(x_\theta^3)) + \varepsilon G_\varepsilon(p_1(t,x_\theta,0)), -\varepsilon L_\varepsilon(P_2(t,x_\theta,0)))$$
$$= O(\varepsilon) \quad \text{for } \varepsilon \to 0$$

We now prove (i). $\forall(w,\mu,y)\in B_\rho$, using (3.3), the above expression for R, (P1) and the fact that $(\int_0^1 y_z^2(t,\xi)\,\mathrm{d}\xi) \le C\|y\|_{E_4}^2$ we deduce that

$$\begin{aligned}\|F_1(w,\mu,y)\| &\le \|-b^{-1}(\varepsilon,\theta)H(\varepsilon,\theta,0,0,0)\| + \|b^{-1}(\varepsilon,\theta)\|\cdot\|R(\varepsilon,\theta,w,\mu,y)\| \\ &\le \bar{C}\varepsilon + \|H_1(\varepsilon,\theta,w,\mu,y) - H_1(\varepsilon,\theta,0,0,0) - b(\varepsilon,\theta)[w,\mu]\| \\ &\le C'(\varepsilon+\varepsilon\rho+\rho^2) \le C''(\varepsilon+\rho^2).\end{aligned}$$

On the other hand, the second component satisfies the inequality $\|F_2(w,\mu,y)\| \le C'''((1/\varepsilon)\rho^3 + \rho^2 + \varepsilon)$ and then

$$|||F(w,\mu,y)||| \le C_7\left(\varepsilon+\rho^2+\frac{\rho^3}{\varepsilon}\right). \tag{3.4}$$

We now prove (ii): $\forall(w,\mu,y),(w',\mu',y')\in B_\rho$ we have

$$\begin{aligned}\|F_1(w,\mu,y) - F_1(w',\mu',y')\| &= \|b^{-1}(\varepsilon,\theta)(R(\varepsilon,\theta,w,\mu,y) - R(\varepsilon,\theta,w',\mu',y'))\| \\ &\le C_8\rho|||(w,\mu,y)-(w',\mu',y')||| \quad \text{and} \\ \|F_2(w,\mu,y) - F_2(w',\mu',y')\| &\le C_8\left(\varepsilon+\rho+\frac{1}{\varepsilon}\rho^2\right)|||(w,\mu,y)-(w',\mu',y')|||.\end{aligned}$$

We need to solve $C_7(\varepsilon+\rho^2+\rho^3/\varepsilon) \le \rho$ and $C_8(\varepsilon+2\rho+(1/\varepsilon)\rho^2) < 1$. These inequalities are solved, for example, choosing $C_6 = 2C_7$ (i.e. $\rho = 2C_7\varepsilon$) and $\varepsilon\in(0,\varepsilon_1)$ with $\varepsilon_1 := \min((4C_7^2(1+2C_7))^{-1};(C_8(1+2C_7)^2)^{-1})$. Then for $\varepsilon\in(0,\varepsilon_1)$ we can apply the contraction mapping theorem in $B_{C_6\varepsilon}$ and then we find a solution $(w(\varepsilon,\theta),\mu(\varepsilon,\theta),y(\varepsilon,\theta))$ with $\|(w(\varepsilon,\theta),\mu(\varepsilon,\theta),y(\varepsilon,\theta))\| \le C_6\varepsilon$, that is Lemma 3.1-(iii). The fact that (w,μ,y) is C^1 is standard, see [7]. □

An immediate consequence of the previous lemma is

Lemma 3.2. *Let* $0<\varepsilon\le\varepsilon_1$. *If* $\mu_\varepsilon(\bar\theta)=0$ *then* $S_\varepsilon(x_{\bar\theta}+w_\varepsilon(\bar\theta),y_\varepsilon(\bar\theta))=0$ *and then* $(x_{\bar\theta}+w_\varepsilon(\bar\theta))\sin(\pi z)+y_\varepsilon(\bar\theta)$ *is an homoclinic solution of system* $(\mathscr{H}_\varepsilon)$.

In the next lemma we give an asymptotic expansion for $\mu_\varepsilon(\theta)$

Lemma 3.3. *Let* $\varepsilon\in(0,\varepsilon_1)$. *Then*

$$\mu_\varepsilon(\theta) = \varepsilon\frac{1}{A}\mathscr{M}(\theta) + O(\varepsilon^2),$$

where

$$A = \int_{\mathbb{R}} \dot{x}_0^2(t)\,\mathrm{d}t$$

and

$$\mathcal{M}(\theta) = \int_{\mathbb{R}} (-p_1(t, x_\theta(t), 0) + \delta \dot{x}_\theta(t))\dot{x}_\theta(t)\, \mathrm{d}t = -\int_{\mathbb{R}} p_1(t, x_\theta(t), 0)\dot{x}_\theta(t)\, \mathrm{d}t + \delta A$$

is the "Melnikov function" of the system. Moreover, $\mathcal{M}(\theta) = \Gamma'(\theta) + \delta A$, *where* Γ *is the Poincaré–Melnikov primitive defined by*

$$\Gamma(\theta) = \int_{\mathbb{R}} W(t, x_\theta(t))\, \mathrm{d}t \tag{3.5}$$

with $-p_1(t, x, 0) = (\mathrm{d}/\mathrm{d}x)W(t, x)$.

Proof. Since $(x_\theta + w_\varepsilon(\theta), y_\varepsilon)$ satisfies Lemma 3.1-(i), $(x_\theta + w_\varepsilon(\theta))$ satisfies the equation

$$-(x_\theta + w_\varepsilon)'' - \varepsilon\delta(x_\theta + w_\varepsilon)' + \beta(x_\theta + w_\varepsilon)$$
$$= \alpha(x_\theta + w_\varepsilon)^3 + \kappa\pi^2(x_\theta + w_\varepsilon)\left(\int_0^1 (\partial_z y_\varepsilon)^2\right) - \varepsilon p_1(t, x_\theta + w_\varepsilon, y_\varepsilon) - \mu_\varepsilon(\theta)\dot{x}_\theta. \tag{3.6}$$

Multiplying (3.6) by $\dot{x}_\theta$, integrating on $\mathbb{R}$, using that, for Lemma 3.1-(iii), $(w_\varepsilon, y_\varepsilon) = O(\varepsilon)$ and that $-\ddot{x}_\theta + \beta x_\theta = \alpha x_\theta^3$ we deduce that

$$\int_{\mathbb{R}} (-w_\varepsilon'' + \beta w_\varepsilon - \alpha 3x_\theta^2 w_\varepsilon)\dot{x}_\theta + O(\varepsilon^2) + \varepsilon\mathcal{M}(\theta) = \mu_\varepsilon(\theta)\int_{\mathbb{R}} \dot{x}_0^2.$$

Integrating by parts the first integral and using equation $-(\dot{x}_\theta)'' + \beta\dot{x}_\theta = 3\alpha x_\theta^2 \dot{x}_\theta$ we deduce Lemma 3.3.

4. Existence of homoclinic solutions of ($\mathcal{H}\varepsilon$)

By Lemmas 3.2 and 3.3, it follows that the existence of "topologically simple" zeroes of the Melnikov function $\mathcal{M}(\theta)$ implies the existence of homoclinic solutions of system ($\mathcal{H}_\varepsilon$).

Definition 4.1. We say that the Melnikov function $\mathcal{M}$ possesses in the interval $(\bar{\theta} - R, \bar{\theta} + R)$ a "topologically simple" zero if $\mathcal{M}(\bar{\theta} - R) \cdot \mathcal{M}(\bar{\theta} + R) < 0$, i.e. if $\mathcal{M}$ changes sign on $[\bar{\theta} - R, \bar{\theta} + R]$.

Remark 4.1. We underline that the Melnikov function $\mathcal{M}$ always possesses zeroes "topologically simple" when the perturbation $P(\cdot, w)$ is periodic, quasi-periodic or almost periodic in time, if the damping term is not too large and $\mathcal{M}(\theta)$ is non-constant. Indeed, in the former cases $\mathcal{M}$ has infinitely many "topologically simple" zeroes: the condition "Melnikov oscillating" defined below is always satisfied. Indeed, in these cases the Poincaré–Melnikov function Γ is resp. periodic, quasi-periodic, almost periodic (see [7]) and then it is easy to see that $\Gamma'(\theta)$ satisfies condition "Melnikov oscillating". For δ small enough the same holds for $\mathcal{M}(\theta)$.

Theorem 4.1. *Assume* (P1) *and* (γ1). *If the Melnikov function has a topologically simple zero for some* $\bar{\theta} \in \mathbb{R}$ *then for* ε *small enough system* ($\mathscr{H}_\varepsilon$) *has an homoclinic solution* u_ε *near* $x_0(\cdot - \tilde{\theta})\sin(\pi z)$ *with* $\tilde{\theta} \in (\bar{\theta} - R, \bar{\theta} + R)$.

It is now possible, reasoning as in [7], to build multibump homoclinic solutions leading to the existence of a chaotic dynamics. Let assume

Condition "Melnikov oscillating": *There are* $\bar{m} > 0$ *and a sequence* $\{U_n=(c_n, d_n)\}_{n \in \mathbb{Z}}$ *of bounded open intervals of* $\mathbb{R}$ *which satisfy*:
(i) *For any* $n \in \mathbb{Z}$ "$\mathscr{M}(c_n) > \bar{m}$ *and* $\mathscr{M}(d_n) < -\bar{m}$" *or* "$\mathscr{M}(c_n) < -\bar{m}$ *and* $\mathscr{M}(d_n) > \bar{m}$".
(ii) $c_n \to +\infty$ *as* $n \to +\infty$ *and* $d_n \to -\infty$ *as* $n \to -\infty$.
We can prove the existence of two-bumps homoclinics

Theorem 4.2. *Let* (P1), (γ1) *and condition* "*Melnikov oscillating*" *hold. There exists* $\varepsilon_2 > 0$ *such that* $0 < \varepsilon \le \varepsilon_2$ *and there exists* D_ε *such that if* $c_{i_2} - d_{i_1} > D_\varepsilon$ *then there exists a homoclinic solution* u_ε *located near some* $(x_{\theta_1} + x_{\theta_2})\sin(\pi z)$ *with* $\theta_1 \in U_{i_1}$ *and* $\theta_2 \in U_{i_2}$.

Because of the exponential decay at infinity of x_θ the existence of solutions with infinitely many bumps follows.

Theorem 4.3. *Let* (P1), (γ1) *and condition* "*Melnikov oscillating*" *hold.* $\forall \rho > 0$ *there is* $\varepsilon_3 > 0$ *such that* $\forall 0 < \varepsilon \le \varepsilon_3$, *there exists* $\bar{D}_\varepsilon > 0$ *such that for any sequence of intervals* $(U_{i_l} = (c_{i_l}, d_{i_l}))_{l \in J \subset \mathbb{Z}}$ *satisfying* $\inf_{l \in J}(c_{i_{l+1}} - d_{i_l}) > D_\varepsilon$, *there are* $(\theta_l)_{l \in J}$ *with* $\theta_l \in U_{i_l} = (c_{i_l}, d_{i_l})$ *and a solution* u_ε *of* ($\mathscr{H}_\varepsilon$) *which satisfies*

$$\left\| u_\varepsilon - \sum_{l \in J} x_{\theta_l} \sin(\pi z) \right\|_{L^\infty(\mathbb{R}, C^4_D)} \le \rho.$$

If J *is infinite, such a solution* u_ε *has infinitely many bumps.*

The last theorem implies that the topological entropy of the system is positive (see [7,12]).

5. Other applications

In this section, we consider the following two cases:
(i) The deflection of the beam $\mathbf{w}$ lies in a N-dimensional ($N \ge 2$) space;
(ii) $m^2\pi^2 < \gamma < (m+1)^2\pi^2$ ($m \ge 2$) and then the equilibrium $w = 0$ has m-dimensional stable and unstable manifolds.
In both cases (i) and (ii), system ($\mathscr{H}_\varepsilon$) is no longer a perturbation of a planar system and then the techniques of [10] cannot be applied.

5.1. Radial systems

We consider the following system of PDEs:

$$(\mathscr{H}_\varepsilon) \quad \mathbf{w}_{tt} + \mathbf{w}_{zzzz} + \gamma \mathbf{w}_{zz} - \kappa \left(\int_0^1 |\mathbf{w}_z(t,\xi)|^2 \, \mathrm{d}\xi \right) \mathbf{w}_{zz} = \varepsilon(\mathbf{P}(t,\mathbf{w}) - \delta \mathbf{w}_t),$$

where $\mathbf{w} = (w^1, \ldots, w^N) \in \mathbb{R}^N$, $N \geq 2$, with boundary conditions $\mathbf{w}(t,0) = \mathbf{w}(t,1) = \mathbf{w}_{zz}(t,0) = \mathbf{w}_{zz}(t,1) = 0$.

For any integer $k \in \mathbb{N}$ we define as in Section 2 the spaces

$$C_D^k(I,\mathbb{R}^N) = \left\{ \mathbf{u}(z) = \sum_{j \geq 1} \mathbf{u}_j \sin(j\pi z) \,\middle|\, \sum_{j \geq 1} |\mathbf{u}_j| j^k < +\infty \right\},$$

$$\mathbf{u}_j = (u_j^1, \ldots, u_j^N) \in \mathbb{R}^N, \quad |\mathbf{u}| = \sqrt{\sum_{i=1}^N (u^i)^2},$$

with norm $\|\mathbf{u}\|_{C_D^k} = \sum_{j \geq 1} |\mathbf{u}_j| j^k$.

Similarly, we consider the spaces of curves

$$E_k = \left\{ \mathbf{w}(t,z) = \sum_{j \geq 1} \mathbf{w}_j(t) \sin(j\pi z) \,\middle|\, \mathbf{w}_j(\cdot) \in C_0(\mathbb{R},\mathbb{R}^N) \text{ and } \sum_{j \geq 1} \|\mathbf{w}_j\|_\infty j^k < +\infty \right\}$$

with norm $\|\mathbf{w}\|_{E_k} = \sum_{j \geq 1} \|\mathbf{w}_j\|_\infty j^k$.

Assuming ($\gamma 1$), the equilibrium solution $\mathbf{w}=0$ of $(\mathscr{H}_0)$ has N-dimensional stable and unstable manifolds $W_0^{\mathrm{s,u}}$, they coincide and, due to the $SO(N)$-invariance of $(\mathscr{H}_0)$, $W_0^{\mathrm{s,u}}$ are filled by the homoclinics $\xi x_\theta(t)$, where $\xi \in S^{N-1}$, $\theta \in \mathbb{R}$ and $x_\theta = x_0(\cdot - \theta)$ are the homoclinics of the scalar problem

$$-\ddot{x} + \beta x = \alpha x^3.$$

In other words, $(\mathscr{H}_0)$ possesses an N-dimensional manifold of homoclinics to 0

$$Z = \{\xi x_\theta(\cdot) \colon \theta \in \mathbb{R}, \ \xi \in S^{N-1}\}.$$

Z is diffeormorphic to $\mathbb{R} \times S^{N-1}$ and its tangent space is $T_{\xi x_\theta} Z = \{\mu \xi \dot{x}_\theta + \eta x_\theta \colon \mu \in \mathbb{R}, \eta \in T_\xi S^{N-1}\}$.

Remark 5.1. It results that $T_{\xi x_\theta} Z = \mathscr{K}$, where $\mathscr{K}$ is the linear space of the solutions $\mathbf{v}$ of the linear system $-\ddot{\mathbf{v}} + \beta \mathbf{v} = 3\alpha x_\theta^2 \mathbf{v}$ with $\mathbf{v}(t) \to 0$ as $|t| \to +\infty$. Indeed since Z ia a N-dimensional manifold of homoclinic orbits, by differentiation one has that $T_{\xi x_\theta} Z \subset \mathscr{K}$. Moreover, $W_0^{\mathrm{s}} = W_0^{\mathrm{u}}$ and $\dim W_0^{\mathrm{u,s}} = N$. For any $p \in W_0^{\mathrm{s}} = W_0^{\mathrm{u}}$ $\dim \mathscr{K} = \dim T_p W_0^{\mathrm{s}} \cap T_p W_0^{\mathrm{u}} = N$, (see [2]). We conclude that $T_{\xi x_\theta} Z = \mathscr{K}$.

We require that the perturbation satisfies the analogue of hypothesis (P1)

(P1′) $\mathbf{P}(t,\mathbf{w}) \in C^1(\mathbb{R} \times C_D^4(I,\mathbb{R}^N), C_D^2(I,\mathbb{R}^N))$ with $\mathbf{P}(t,\mathbf{0})=0$, $D_{\mathbf{w}}\mathbf{P}(t,\mathbf{0})=0$. $\mathbf{P}(\cdot,\mathbf{w})$, $D_{\mathbf{w}}\mathbf{P}(\cdot,\mathbf{w}) \in L^\infty(\mathbb{R})$ on bounded sets of C_D^4 and there exists $\rho_0 > 0$ such that in $B(0,\rho_0)=\{\mathbf{w}\in C^4_D \mid \|\mathbf{w}\|_{C^4_D}\le\rho_0\}$ $D_{\mathbf{w}}P(t,\mathbf{w})$ is Λ-Lipschitz continuous, i.e. for all $\mathbf{w},\bar{\mathbf{w}}\in B(0,\rho_0)$.

Then $\mathbf{w}\equiv 0$ is an equilibrium still for $(\mathscr{H}_\varepsilon)$, the linearized system in 0 is formally identical to $(\mathscr{L}_\varepsilon)$ and hence has the same eigenvalues λ_j^ε, given by (2.1), which now have geometric multiplicity N.

As in Section 2, we split $\mathbf{w}$ as

$$\mathbf{w}(t,z)=\mathbf{x}(t)\sin(\pi z)+\mathbf{u}(t,z)$$

with $\mathbf{u}(t,z)=\sum_{j\ge 2}\mathbf{u}_j(t)\sin(j\pi z)$. We get a system formally identical to $(\mathscr{S}_\varepsilon)$

$$(\mathscr{S}_\varepsilon)\quad\begin{cases} -\ddot{\mathbf{x}}-\varepsilon\delta\dot{\mathbf{x}}+\beta\mathbf{x}=\alpha|\mathbf{x}|^2\mathbf{x}+\kappa\pi^2\mathbf{x}\left(\displaystyle\int_0^1|\mathbf{u}_z(t,\xi)|^2\,\mathrm{d}\xi\right)-\varepsilon\mathbf{p}_1(t,x,u),\\ \mathbf{u}_{tt}+\varepsilon\delta\mathbf{u}_t+\mathbf{u}_{zzzz}+(\gamma-\frac{\kappa|\mathbf{x}|^2\pi^2}{2})\mathbf{u}_{zz}=\kappa\left(\displaystyle\int_0^1|\mathbf{u}_z(t,\xi)|^2\,\mathrm{d}\xi\right)\mathbf{u}_{zz}+\varepsilon\mathbf{P}_2(t,x,u)\end{cases}$$

the only difference being that the the first component is an N-dimensional system while the second one is an N-dimensional system of PDEs. The extension to this case of the procedure explained in Section 2 is quite easy and we get

Lemma 5.1. *There are* ε_4, $C_9>0$ *and smooth functions* $(\mathbf{w}(\varepsilon,\theta,\xi),\mathbf{y}(\varepsilon,\theta,\boldsymbol{\xi}),(\mu,\eta)(\varepsilon,\theta,\xi)):(-\varepsilon_0,\varepsilon_0)\times\mathbb{R}\times S^{N-1}\to C_0(\mathbb{R},\mathbb{R}^N)\times E_4\times\mathbb{R}\times TS^{N-1}$ *such that*

(i) $S_\varepsilon(\xi x_\theta+\mathbf{w}_\varepsilon(\theta,\xi),\mathbf{y}_\varepsilon(\theta,\xi))=(G_\varepsilon(\mu_\varepsilon(\theta,\xi)\xi\dot{x}_\theta+\eta_\varepsilon(\theta,\xi)x_\theta),0)$;

(ii) $(\mathbf{w}_\varepsilon(\theta,\xi),\xi\dot{x}_\theta)_{L^2}=0$ *and* $\int_{\mathbb{R}}(\mathbf{w}-(\mathbf{w}\cdot\xi)\xi)x_\theta\,\mathrm{d}t=0$;

(iii) $\|\mathbf{w}_\varepsilon(\theta,\xi)\|_\infty,\|\mathbf{y}_\varepsilon(\theta,\xi)\|_{E_4}\le C_9\varepsilon$ *for all* $0<\varepsilon\le\varepsilon_4$, *for all* $\theta\in\mathbb{R}$, $\xi\in S^{N-1}$.

Proof. We define the function $H=(H_1,H_2):\mathbb{R}\times\mathbb{R}\times S^{N-1}\times C_0\times E_4\to\mathbb{R}^N\times C_0\times E_4$ defined by

$$H_1=\begin{pmatrix}\xi(\mathbf{w},\xi\dot{x}_\theta)_{L^2}+\displaystyle\int_{\mathbb{R}}(\mathbf{w}-(\mathbf{w}\cdot\xi)\xi)x_\theta\,\mathrm{d}t\\ \mathbf{w}-\alpha(G_\varepsilon(|\xi x_\theta+\mathbf{w}|^2(\xi x_\theta+\mathbf{w})-G_0(\xi x_\theta^3))-G_\varepsilon\left(\kappa\pi^2(\xi x_\theta+\mathbf{w})\displaystyle\int_0^1|\mathbf{y}_z|^2-\varepsilon\mathbf{p}_1\right)\\ -G_\varepsilon(\mu\xi\dot{x}_\theta+\eta x_\theta)\end{pmatrix},$$

$$H_2=\left(\mathbf{y}-L_\varepsilon\left(\frac{\kappa\pi^2}{2}(2\xi x_\theta\mathbf{w}+\mathbf{w}^2)\mathbf{y}_{zz}+\kappa\left(\int_0^1|\mathbf{y}_z|^2\right)\mathbf{y}_{zz}+\varepsilon\mathbf{P}_2(t,\xi x_\theta+\mathbf{w},\mathbf{y})\right)\right).$$

The operator

$$\frac{\partial H_1}{\partial(\mathbf{w},\mu,\eta)}_{|(\varepsilon,\theta,\xi,0,0,0)}[\mathbf{w},\mu,\eta]$$

$$=\begin{pmatrix}\xi(\mathbf{w},\xi\dot{x}_\theta)_{L^2}+\displaystyle\int_{\mathbb{R}}(\mathbf{w}-(\mathbf{w}\cdot\xi)\xi)x_\theta\,\mathrm{d}t\\ \mathbf{w}-G_\varepsilon(3\alpha x_\theta^2\mathbf{w})+\varepsilon G_\varepsilon(\partial_x\mathbf{p}_1(t,x_\theta,0)\mathbf{w})-G_\varepsilon(\mu\xi\dot{x}_\theta+\eta x_\theta)\end{pmatrix}\tag{5.1}$$

is "Id + Compact". Setting $b(\varepsilon,\theta,\xi)=\partial H/\partial(\mathbf{w},\mu)_{|(\varepsilon,\theta,\xi,0,0,0)} \in \mathscr{L}(C_0(\mathbb{R})\times\mathbb{R})$ the operator $b(0,\theta,\xi)$ is injective (and hence invertible). In fact, the identity $T_{\xi x_\theta} Z = \mathscr{K}$ holds as a consequence of Remark 5.1. Now, the proof follows Lemma 3.1. □

Reasoning as in Lemma 3.3 one finds that $\mu(\varepsilon,\theta,\xi) = \varepsilon(1/A)\mathscr{M}_1(\theta,\boldsymbol{\xi}) + O(\varepsilon^2)$ and $\eta(\varepsilon,\theta,\xi)=\varepsilon(1/B)\mathscr{M}_2(\theta,\xi)+O(\varepsilon^2)$ where $B=\int_{\mathbb{R}} x_0^2\,\mathrm{d}t$ and $\mathscr{M}=(\mathscr{M}_1(\theta,\boldsymbol{\xi}),\mathscr{M}_2(\theta,\xi))$: $\mathbb{R}\times S^{N-1} \to \mathbb{R}\times TS^{N-1}$

$$\mathscr{M}_1(\theta,\xi) = \int_{\mathbb{R}} -\mathbf{p}_1(t,\xi x_\theta(t),0)\xi\dot{x}_\theta(t)\,\mathrm{d}t + \delta A,$$

$$\mathscr{M}_2(\theta,\xi) = proj_{T_\xi S^{N-1}} \int_{\mathbb{R}} -\mathbf{p}_1(t,\xi x_\theta(t),0)x_\theta(t)\,\mathrm{d}t$$

is the "Melnikov vector field". "Topologically simple" zeroes of $\mathscr{M}$ give rise to homoclinic solutions of $(\mathscr{H}_\varepsilon)$.

If $\mathbf{p_1}(t,\mathbf{w}) = \nabla_{\mathbf{w}}F(t,\mathbf{w})$ (namely, if the perturbation $\mathbf{p}_1$ on the first mode is Hamiltonian) we get that $\mathscr{M}(\theta,\xi) = \nabla\Gamma(\theta,\xi) + \delta A\mathrm{e}_1$, where $\Gamma(\theta,\xi) = \int_{\mathbb{R}} F(t,\xi x_\theta(t))\,\mathrm{d}t$ is the Poincaré–Melnikov primitive of the system. Thus, topologically non-degenerate critical points $(\bar{\xi},\bar{\theta})$ of Γ give rise to topologically simple zeroes of $\mathscr{M}$ if δ is sufficiently small. The existence of a chaotic dynamics follows as in [7].

5.2. Greater values of γ

In this section, we assume that (γ2) the load γ satisfies $m^2\pi^2 < \gamma < (m+1)^2\pi^2$ for $m \in \mathbb{N}$ and $m \geq 2$.

Assuming (γ2), the equilibrium solution $w=0$ has m positive and m negative eigenvalues and still possesses an infinite dimensional center manifold. Looking for solutions of $(\mathscr{H}_0)$ like

$$w(t,z) = \sum_{j=1}^{m} x_j(t)\sin(j\pi z)$$

we obtain that $\mathbf{x} = (x_1,\ldots,x_m)$ satisfies the following Hamiltonian system with Hamiltonian R_0 (the m-mode Galierkin approximation of $(\mathscr{H}_0)$):

$$(\mathscr{R}_0) \qquad \begin{cases} -\ddot{x}_1 + \lambda_1^2 x_1 = \alpha x_1 \left(\sum_{l=1}^{m} l^2 w_l^2\right), \\ \ldots \\ -\ddot{x}_j + \lambda_j^2 x_j = \alpha j^2 x_j \left(\sum_{l=1}^{m} l^2 w_l^2\right) \quad 2 \leq j \leq m-1, \quad \alpha = \dfrac{\kappa\pi^4}{2}, \\ \ldots \\ -\ddot{x}_m + \lambda_m^2 x_m = \alpha m^2 x_m \left(\sum_{l=1}^{m} l^2 w_l^2\right). \end{cases}$$

The functions $(0,\ldots,\pm x_{j,\theta},\ldots,0)$ are homoclinic solutions to 0 of system $(\mathscr{R}_0)$, where $x_{j,\theta}(t) = x_{j,0}(t-\theta)$ and $x_{j,0}= \sqrt{4\lambda_j^2/\kappa\pi^4 j^4}\,\mathrm{sech}(\lambda_j t)$ is the homoclinic solution of the

Duffing equation $-\ddot{x}+\lambda_j^2 x=\alpha j^4 x^3$ with $\dot{x}_{j,0}(0)=0$ and $x_{j,0}(t)>0$. In other words, the m-dimensional stable and unstable manifolds $W_0^{\mathrm{s,u}}$ of $w=0$ intersect at least along the homoclinic orbits $u_{j,\theta}=\pm x_{j,\theta}(t)\sin(j\pi z)$.

Then the unperturbed equation $(\mathscr{H}_0)$ possesses at least $2m$ families of homoclinics. In order to apply the reduction approach of Section 3, one needs to check that one of these homoclinics is "transversal on the zero energy level" according to the following definition (see [5]).

Definition 5.1. An homoclinic solution $\mathbf{x}_0(t)$ of $(\mathscr{R}_0)$ is said "transversal on the zero energy level $\{R_0(\mathbf{x})=0\}$" if $W_0^{\mathrm{s,u}}$ intersect along $(\mathbf{x}_0(t),\dot{\mathbf{x}}_0(t))_{t\in\mathbb{R}}$ transversally on $\{R_0(\mathbf{x})=0\}$.

This is the case for $x_{m,\theta}\sin(m\pi z)$:

Lemma 5.2. *The homoclinic solution* $(0,\dots,x_{m,\theta})$ *of* $(\mathscr{R}_0)$ *is "transversal on the zero energy level* $\{R_0(\mathbf{x})=0\}$*".*

Proof. In [5], it is shown that this condition is equivalent to require that the unique solution $(v_1,\dots,v_m)$ of the linear system

$$\begin{aligned}&-\ddot{v}_j+\lambda_j^2 v_j-(\alpha j^2 m^2 x_{m,\theta}^2)v_j=0,\quad 1\le j\le m-1,\\&-\ddot{v}_m+\lambda_m^2 v_m-(3\alpha m^4 x_{m,\theta}^2)v_m=0\end{aligned}\tag{5.2}$$

with $v_j(t)\to 0$ for $|t|\to+\infty$ $(1\le j\le m)$ is, up to a multiplicative factor, $(0,\dots,\dot{x}_{m,\theta})$. This last assertion is a simple consequence of the following lemma

Lemma 5.3. *Let* $u:\mathbb{R}\to\mathbb{R}$ *be a bounded solution of the linear equation*

$$-\ddot{u}+u-\phi(t)u=0,\tag{5.3}$$

where $\phi(t)$ *is a continuous function satisfying* $\lim_{t\to\pm\infty}\phi(t)=0$. *If either one of the following conditions is satisfied*

(i) $\phi(t)\le 1\ \forall t\in\mathbb{R}$;

(ii) $(\int_{\mathbb{R}}\phi^2(t)\,\mathrm{d}t)^{1/2}<4/\sqrt{3}$,

then $u\equiv 0$ *is the unique bounded solution of* (5.3).

Using Lemma 5.3 we show how it can be used to prove Lemma 5.2. The equations forming system (5.2) are decoupled. It is easy to show that $\dot{x}_{\theta,m}$ is the unique bounded solution of the mth equation. Then we just have to check that the unique bounded solution of the first $(m-1)$ equations is the trivial one. In order to do this, we perform the rescaling

$$u_j(t)=v_j(t/\lambda_j),\quad j=1,\dots,m-1$$

so that (5.2) is equivalent to

$$-\ddot{u}_j+u_j-\phi_j(t)u_j=0\quad\text{with } \phi_j(t)=\frac{\alpha j^2 m^2}{\lambda_j^2}x_{m,\theta}^2(t/\lambda_j)$$

so it is easy to check, using the explicit expression of $x_{m,\theta}$, that

$$\|\phi_j\|_\infty \le 2\frac{j^2}{m^2}\frac{\lambda_m^2}{\lambda_j^2} \quad \text{and} \quad \left(\int_{\mathbb{R}} \phi_j^2(t)\,\mathrm{d}t\right)^{1/2} = 2\frac{j^2}{m^2}\frac{\lambda_m^2}{\lambda_j^2}\sqrt{\frac{\lambda_j}{\lambda_m}\frac{4}{3}}.$$

Using the first estimate we see that condition (i) of the lemma is satisfied for all $j < m-1$ while using the second estimate we conclude that condition (ii) is satisfied for $j = m-1$. □

It is possible to perform the finite-dimensional reduction of the previous section obtaining the following Melnikov function:

$$\mathscr{M}^m(\theta) = -\int_{\mathbb{R}} p_m(t, u_{m,\theta}(t))\dot{x}_{m,\theta}(t)\,\mathrm{d}t + \delta\int_{\mathbb{R}} \dot{x}_{m,0}^2(t)\,\mathrm{d}t.$$

The existence of homoclinics and a chaotic dynamics for $(\mathscr{H}_\varepsilon)$ follow

Theorem 5.1. *Let* (P1), (γ2) *hold. Assume that* $\mathscr{M}^m$ *satisfies condition "Melnikov oscillating". Then the same statement of Theorem* 4.3 (*where* $x_\theta \sin(\pi z)$ *is replaced by* $x_{m,\theta}\sin(m\pi z)$) *holds.*

6. Homoclinics to small non-constant trajectories

In Section 2 with assumption (P1), we have required that $w=0$ remains an equilibrium solution for system $(\mathscr{H}_\varepsilon)$, $\varepsilon \neq 0$. This of course rules out perturbations like those considered by Holmes and Marsden in [10], namely $P(t, w(t,z)) = f(z)\cos\omega t$, like all the ones that do not depend on w. Nevertheless for this particular perturbation it can be shown that, when $\omega \neq |\lambda_j|$ for $j = 2, 3, \ldots$ for ε small enough there exists a unique $u^\varepsilon \in E_4$ solution of $(\mathscr{H}_\varepsilon)$ with $\|u^\varepsilon\|_{E_4} = O(\varepsilon)$ which bifurcate from the unperturbed equilibrium $w=0$. Therefore, one looks for a solution of $(\mathscr{H}_\varepsilon)$ homoclinic to u^ε namely a solution $w^\varepsilon \in E_4$ such that $\|w^\varepsilon(t,\cdot) - u^\varepsilon(t,\cdot)\|_{C^4_D(I)} \to 0$ and $\|\dot{w}^\varepsilon(t,\cdot) - \dot{u}^\varepsilon(t,\cdot)\|_{C^4_D(I)} \to 0$ as $|t| \to +\infty$.

For hyperbolic equilibrium states the existence of such solutions u_ε which bifurcate from the equilibrium is standard. In the present case, dealing with an equilibrium with an infinite set of pure imaginary eigenvalues, this is not always true. We need to avoid the resonant cases between the forcing frequencies and such eigenvalues (the frequencies of the small oscillations near the equilibrium). To be more precise let us state some lemmas.

Lemma 6.1. *There is a* $C_{10} > 0$ *such that for any* $f \in B_2$ *there exists a unique solution* $h := L_\varepsilon(f) \in B_4$ *of*

$$h_{tt} + \varepsilon\delta h_t + h_{zzzz} + \gamma h_{zz} = f.$$

The linear operator $L^\varepsilon : B_2 \to B_4$ *is continuous and satisfies the condition* $\|L^\varepsilon\|_{\mathscr{L}(B_2,B_4)} \le C_{10}/\varepsilon\delta$. *Moreover, if* f *is almost periodic* (*periodic*) *in time, also* h *is almost periodic* (*periodic*) *in time.*

Proof. The proof of the lemma is obtained decomposing $f(t,z)=\sum_{j\geq 1} f_j(t)\sin(j\pi z)$ and $u(t,z)=\sum_{j\geq 1} u_j(t)\sin(j\pi z)$. □

Now, we assume that (P2) $P(t,w)=P(t)$ with $P\in B_2$, namely, $P(t)(z)=\sum_{j=1}^{+\infty} p_j(t)\sin(j\pi z)$ with $\sum_{j=1}^{+\infty} j^2\|p_j(\cdot)\|_\infty < +\infty$.

By Lemma 6.1, we see that although the equation

$$h_{tt}+\varepsilon\delta h_t+h_{zzzz}+\gamma h_{zz}=\varepsilon P(t) \tag{6.1}$$

has, for all $\varepsilon>0$, a unique solution $h^\varepsilon\in B_4$, it is not always true that $\|h^\varepsilon\|_{B_4}\to 0$ as $\varepsilon\to 0$ (this happens when in the perturbation $P(t)$ there are frequencies in resonance with the $|\lambda_j|$).

Thus, we make the following assumptions in order to avoid such resonant cases:

(Ri) There exists $C_{11}>0$ such that the solutions of (6.1) satisfy $\|h^\varepsilon\|_{B_4}\leq C_{11}\varepsilon$ when $\varepsilon\to 0$;

(Ri′) there is $v\in B_4$ such that $\|h^\varepsilon-\varepsilon v\|_{B_4}=O(\varepsilon^2)$.

Condition (Ri′) clearly implies (Ri).

Remark 6.1. Both conditions (Ri) and (Ri′) are satisfied in the case considered by Holmes and Marsden, namely, when $P(t)(z)=f(z)\cos\omega t$ if $\omega^2\neq j^2\pi^2(j^2\pi^2-\gamma)$ for all $j\geq 2$. More generally, the same computations and the superposition principle show that also perturbations like $P(t)(z)=\sum_{k=1}^{N} f_k(z)\cos(\omega_k t+\theta_k)$ satisfy (Ri) and (Ri′) as soon as $\omega_k^2\neq j^2\pi^2(j^2\pi^2-\gamma)$ for all $j\geq 2$ and $1\leq k\leq N$; moreover, if some ratio $\omega_k/\omega_{k'}$ is not rational we shall get a solution which (as the forcing) is quasi periodic but not periodic. One advantage of our approach is that we can still construct a Melnikov function, even if the forcing is not periodic.

We now consider solutions of the non-linear system $(\mathscr{H}_\varepsilon)$.

Lemma 6.2. *Assume* (Ri). *Then there is* $\varepsilon_5>0$ *such that* $\forall\varepsilon\in(0,\varepsilon_5)$ *there is a unique* $u^\varepsilon\in B_4$ *solution of* $(\mathscr{H}_\varepsilon)$ *with* $\|u^\varepsilon-h^\varepsilon\|_{B_4}\leq\varepsilon$. *More precisely, we have* $\|u^\varepsilon-h^\varepsilon\|=O(\varepsilon^2)$ *as* $\varepsilon\to 0$. *Moreover, if* $P(t)$ *is almost periodic in time, then* u^ε *is almost periodic in time.*

Proof. The proof is obtained once again using the contraction mapping theorem. Define the operator $N: C_D^4(I)\to C_D^2(I)$ by $N(u):=(\kappa\int_0^1 u_z^2\,\mathrm{d}\zeta)u_{zz}$. It is easy to check that it induces an operator (that we shall call in the same way) $N:B_4\to B_2$ which is homogeneous of degree 3; moreover,

$$\|N(u)\|_{B_2}\leq C_{11}\|u\|_{B_4}^3 \quad \|dN(u)\|_{\mathscr{L}(B_4,B_2)}\leq C_{11}\|u\|_4^2.$$

We look for a solution of $(\mathscr{H}_\varepsilon)$ of the form $u^\varepsilon=w+h^\varepsilon$; plugging it into equation $(\mathscr{H}_\varepsilon)$ we see that u^ε must satisfy equation $(\mathscr{H}_\varepsilon)$ iff the fixed-point problem $w=L^\varepsilon(N(h^\varepsilon+w))$ has a solution $w\in B_4$. Setting $\Phi^\varepsilon(w):=L^\varepsilon\circ N(h^\varepsilon+w)$ we see that $\Phi^\varepsilon:B_4\to B_4$. We claim that there is an $\varepsilon_5>0$ such that

(i) $\Phi^\varepsilon(B_\varepsilon)\subset B_\varepsilon$, for all $\varepsilon\in(0,\varepsilon_5)$;

(ii) Φ^ε is a contraction on B_ε for all $\varepsilon\in(0,\varepsilon_5)$.

The proof is simple: if $\|w\|_{B_4} \le \varepsilon$ then

$$\|\Phi^\varepsilon(w)\|_{B_4} = \left\| \Phi^\varepsilon(0) + \int_0^1 \mathrm{d}\Phi^\varepsilon(sw)[w]\,\mathrm{d}s \right\|_{B_4} \le \varepsilon^2 C_{12}. \tag{6.2}$$

Moreover, we see that, if $v, w \in B_\varepsilon$, then

$$\|\Phi^\varepsilon(w) - \Phi^\varepsilon(v)\|_{B_4} = \left\| \int_0^1 \mathrm{d}\Phi^\varepsilon(v + s(w-v))[w-v]\,\mathrm{d}s \right\|_{B_4} \le \varepsilon C_{13} \|w - v\|_{B_4}.$$

Choosing $\varepsilon_5 < \min\{1/C_{12}, 1/C_{13}\}$ both conditions are fulfilled for $0 < \varepsilon < \varepsilon_5$ and the claim is proved. By the contraction mapping theorem we get a unique solution w^ε in B_ε. On the other hand, since $w^\varepsilon = \Phi^\varepsilon(w^\varepsilon)$ by Eq. (6.2) $w^\varepsilon = u^\varepsilon - h^\varepsilon$ is $O(\varepsilon^2)$.

As far as the almost periodicity (or periodicity) is concerned, we can check directly that if $P(t)$ is uniformly almost periodic (periodic) then also each component h_j^ε, the solution of the linear problem, is almost periodic (periodic) and hence, by uniform convergence, $h^\varepsilon(t,\cdot)$ is uniformly almost periodic (periodic). To deduce that also u^ε is almost periodic we may perform the previous contraction mapping on the subspace $\tilde{B} = \{w \in B_4 \colon w = \sum w_j(t)\sin(\pi j z);\ w_j \text{ almost periodic (periodic) } \forall j \in \mathbb{N}\}$. □

Since we are interested in solutions homoclinic to u^ε we write

$$(u^\varepsilon + w)_{tt} + \varepsilon\delta(u^\varepsilon + w)_t + (u^\varepsilon + w)_{zzzz} + \gamma(u^\varepsilon + w)_{zz} + N(u^\varepsilon + w) = P(t), \quad w \in E_4$$

with $N(u) := (\kappa \int_0^1 u_z^2\,\mathrm{d}\xi) u_{zz}$. Since u^ε is a solution of $(\mathscr{H}_\varepsilon)$ the last equation can be written as

$$w_{tt} + \varepsilon\delta w_t + w_{zzzz} + \gamma w_{zz} + N(w) = N(u^\varepsilon) + N(w) - N(u^\varepsilon + w). \tag{6.3}$$

If we assume hypothesis (Ri′) then

$$N(u^\varepsilon) + N(w) - N(u^\varepsilon + w) = \varepsilon P(t,w) + Q(\varepsilon, t, w),$$

where

$$P(t,w) = -\mathrm{d}N(w)[v(t,\cdot)] = \left(-\kappa \int_0^1 2 v_z(t,\xi) w_z(t,\xi)\,\mathrm{d}\xi \right) w_{zz} + \left(-\kappa \int_0^1 w_z^2(t,\xi)\,\mathrm{d}\xi \right) v_{zz} \tag{6.4}$$

is a perturbation which satisfies condition (P1) and $Q(\varepsilon, t, w) = O(\varepsilon^2)$. It is not difficult to check that, even though (due to the term Q) we are not exactly in the same situation of Section 2, the finite-dimensional reduction developed in 3 can be applied. In this case, from (6.4) we deduce that the Melnikov function is

$$\mathscr{M}^*(\theta) = \int_{\mathbb{R}} p_1(t, x_0(t), 0)\dot{x}_\theta + \delta \int_{\mathbb{R}} \dot{x}_0^2(t)\,\mathrm{d}t = \alpha \int_{\mathbb{R}} v_1(t) 3 x_\theta^2(t) \dot{x}_\theta(t)\,\mathrm{d}t + \delta A.$$

Since $-(\dot{x}_\theta)'' + \lambda_1^2 \dot{x}_\theta = 3\alpha x_\theta^2 \dot{x}_\theta$ and $-\ddot{v}_1 + \lambda_1^2 v_1 = p_1(t, x_\theta(t), 0)$, integrating by parts one immediately finds that

$$\mathscr{M}^*(\theta) = \int_{\mathbb{R}} p_1(t, x_\theta, 0)\dot{x}_\theta(t)\,dt + \delta A = \mathscr{M}(\theta),$$

which is the usual Melnikov function. If p_1 is periodic or almost periodic and δ is not too large, $\mathscr{M}$ satisfies the condition "Melnikov oscillating" and then

Theorem 6.1. *Let* (P2), (γ1) *and* (Ri$'$) *hold. Assume that* $\mathscr{M}$ *satisfies condition "Melnikov oscillating". Then, for* ε *small enough, there exist a family of homoclinics to* u_ε *which induce a chaotic behaviour according to Theorem* 4.3.

The same type of result can be obtained using (γ2).

7. Sine–Gordon equation

In this section, we show that the same type of results of the previous sections can be obtained for the following Sine–Gordon equation (see [9])

$$(\mathscr{SG}_\varepsilon) \quad \psi_{tt} - \psi_{zz} + \sin\psi = \varepsilon(P(t,\psi) - \delta\psi_t)$$

with $\psi_z(t,0) = \psi_z(t,1) = 0$ adapting the techniques of the previous sections with those of [5].

We introduce as "phase space" the following Banach spaces of functions of the spatial variable $z \in [0,1] = I$ defined, for any integer $k \in \mathbb{N}$, by

$$C_N^k(I) = \left\{ v(z) = \sum_{j\geq 0} v_j \cos(j\pi z) \;\middle|\; \sum_{j \geq 0} |v_j| j^k < +\infty \right\} \quad \text{with norm}$$

$$\|v\|_{C_N^k} = \sum_{j\geq 0} |v_j| j^k.$$

We define the following spaces of curves:

$$E_k = \Bigg\{ \phi(t,z) = \sum_{j\geq 0} \phi_j(t)\cos(j\pi z) \mid \phi_j(\cdot) \in C_0(\mathbb{R},\mathbb{R}) \text{ (i.e. } \phi_j(t) \to 0 \text{ as } t \to \pm\infty) \text{ and } \sum_{j\geq 0} \|\phi_j\|_\infty j^k < +\infty \Bigg\}$$

with norm $\|\phi\|_{E_k} = \sum_{j\geq 0} \|\phi_j\|_\infty j^k$; moreover, $\tilde{E}_k$ will denote the elements of E_k which have zero mean value in the spatial variable z.

We shall make the following assumption on the perturbation $P(t,\psi)$ (see also Remark 7.1):

(P3) $P(t,\psi) \in C^1(\mathbb{R} \times C_N^4, C_N^2)$ with $P(t, \pm\pi) = 0$, $D_\psi P(t, \pm\pi) = 0$, $P(\cdot,\psi)$, $D_\psi P(\cdot,\psi) \in L^\infty(\mathbb{R})$ on bounded sets of C_N^4 and there exists $\rho_0 > 0$ such that in $B(\pm\pi, \rho_0) = \{\psi \in C_N^4(I) \mid \|\psi - (\pm\pi)\|_{C_N^4(I)} \leq \rho_0\}$ $D_\psi P(t,\psi)$ is Λ-Lipschitz continuous.

It will be useful to write the perturbation P as $P(t,\psi) = p_0(t,\psi) + \sum_{j\geq 1} p_j(t,\psi) \cos(j\pi z) = p_0(t,\psi) + P_1(t,\psi)$, where $P_1(t,\psi) = \sum_{j\geq 1} p_j(t,\psi) \cos(j\pi z) \in \tilde{E}_2$.

By (P3) $\psi(t) = \pm\pi$ are equilibrium solutions of $(\mathscr{SG}_\varepsilon)$, where the linearized equation is

$$(\mathscr{SGL}_\varepsilon) \quad \psi_{tt} + \varepsilon\delta\psi_t - \psi_{zz} - \psi = 0,$$

with $\psi_z(t,0) = \psi_z(t,1) = 0$. Letting $\psi(t,z) = \tilde{\psi}(z)e^{\lambda t}$ and solving for the eigenvalues and eigenvectors we obtain that $\cos(j\pi z)$ for $j = 0,1,\ldots$ are the eigenvectors and that the eigenvalues are

$$\lambda_{\varepsilon,j} = -\frac{\varepsilon\delta}{2} \pm \sqrt{\frac{\varepsilon^2\delta^2}{4} + (1 - j^2\pi^2)}, \quad j = 0,1,2,\ldots\ .$$

An heteroclinic orbit of $(\mathscr{SG}_\varepsilon)$ connecting $-\pi$ to $+\pi$ is a solution $\psi(t,z)$ satisfying

$$\lim_{t\to-\infty} \|\psi(t,\cdot) - (-\pi)\|_{C^4_N} = 0, \quad \lim_{t\to+\infty} \|\psi(t,\cdot) - (+\pi)\|_{C^4_N} = 0, \quad \text{and}$$
$$\lim_{|t|\to+\infty} \|\dot{\psi}(t,\cdot)\|_{C^4_N} \to 0.$$

The unperturbed equation $(\mathscr{SG}_0)$ possesses (on the first mode) two families of heteroclinic solutions

$$x^{\pm}_\theta(t) = \pm 4\arctan\left(\tanh\left(\frac{t-\theta}{2}\right)\right)$$

connecting the equilibrium points $\pm\pi$, namely, $x^+_\theta(t)$ connects $-\pi$ to $+\pi$ and $x^-_\theta(t)$ connects $+\pi$ to $-\pi$.

We will show that for $\varepsilon\delta \neq 0$ small enough $(\mathscr{SG}_\varepsilon)$ has infinitely many solutions $\psi_\varepsilon(t,\cdot)$ winding in the phase space between $\pm\pi$ along the separatrices $x^{\pm}_\theta$.

For simplicity we shall look first for an heteroclinic solution ψ of $(\mathscr{SG}_\varepsilon)$ connecting $-\pi$ to $+\pi$ near some $x^+_\theta(t)$, i.e. $\psi = x^+_\theta + \phi$ with $\phi \in E_4$ small. Clearly, the same computations can be performed for the x^-_θ. From now on we shall simply write x_θ instead that x^+_θ.

It is useful to study system $(\mathscr{SG}_\varepsilon)$ separately along the eigenvector $\cos(0\pi z) = 1$ and the $\varepsilon\delta$-hyperbolic modes $\cos(j\pi z)$ for $j \geq 1$. Then we set

$$\psi(t,z) = x(t) + u(t,z) = \bar{\psi}(t) + u(t,z),$$

where $\bar{\psi}(t) = \int_0^1 \psi(t,z)\,dz$ and $u(t,z) = \psi(t,z) - \bar{\psi}(t) = \sum_{j\geq 1} u_j \cos(j\pi z) \in \tilde{E}_4$ (we then have $\int_0^1 u(z)\,dz = 0$).

Since we look for heteroclinics which branch from the manifold $\{x_\theta\}_{\theta\in\mathbb{R}}$ we write $x(t) = x_\theta + w(t)$ then

$$\psi(t,z) = (x_\theta(t) + w(t)) + u(t,z).$$

Plugging this expression into $(\mathscr{SG}_\varepsilon)$ we obtain an infinite set of second-order equations

$$\begin{aligned}-\ddot{w} - \varepsilon\delta\dot{w} + w = &\int_0^1 \sin(x_\theta + w + u(t,z))\,dz - \sin x_\theta + w(t)\\ &-\varepsilon p_0(t, x_\theta + w + u) + \varepsilon\delta\dot{x}_\theta\end{aligned}$$

$$\ldots$$
$$\ddot{u}_j + \varepsilon\delta\dot{u}_j + (\pi j)^2 u_j - \cos x_\theta(t) = M_j(t,w,u) + \varepsilon p_j(t, x_\theta + w + u), \quad j \geq 1,$$
$$\ldots \tag{7.1}$$

where

$$M_j(t,w,u) = -2\int_0^1 (\sin(x_\theta(t) + w(t) + u(t,z)) + \cos x_\theta(t)) \cos(j\pi z)\,\mathrm{d}z.$$

In compact form, we write

$$\begin{aligned}-\ddot{w} - \varepsilon\delta\dot{w} + w = &\int_0^1 \sin(x_\theta + w + u(t,z))\,\mathrm{d}z - \sin x_\theta) + w(t)\\ &-\varepsilon p_0(t, x_\theta + w + u) + \varepsilon\delta\dot{x}_\theta,\end{aligned}$$

$$u_{tt} + \varepsilon\delta u_t - u_{zz} - (\cos x_\theta)u = M(t,w,u) + \varepsilon P_1(t, x_\theta + w + u), \tag{7.2}$$

where

$$M(t,w,u) = -\sin(x_\theta + w + u) + \int_0^1 \sin(x_\theta + w + u(t,z))\,\mathrm{d}z - \cos x_\theta u.$$

Following the arguments of Section 2, we define the linear Green operators L_ε and G_ε which are, respectively, the inverses of the differential operators

$$\partial_{tt} + \varepsilon\delta\partial_t - \partial_{zz} - (\cos x_\theta) \quad \text{and} \quad -\frac{\mathrm{d}^2}{\mathrm{d}t^2} - \varepsilon\delta\frac{\mathrm{d}}{\mathrm{d}t} + 1,$$

with zero Dirichlet boundary conditions at $t \to \pm\infty$, which allow us to write system $(\mathscr{SG}_\varepsilon)$ in form of integral equations. We can write system (7.2) as $S_{\varepsilon,\theta}(w,u) = 0$ with $S_{\varepsilon,\theta} : C_0(\mathbb{R}) \times \tilde{E}_4 \to C_0(\mathbb{R}) \times \tilde{E}_4$ defined by

$$S_{\varepsilon,\theta}(w,u) = \begin{pmatrix} w - G_\varepsilon\left(\displaystyle\int_0^1 \sin(x_\theta + w + u)\,\mathrm{d}z + w - \sin(x_\theta)\right.\\ \left. -\varepsilon p_0(t, x_\theta + w + u) + \varepsilon\delta\dot{x}_\theta\right) \\ u - L_\varepsilon(M(t,w,u) + \varepsilon P_1(t, x_\theta + w + u)) \end{pmatrix}. \tag{7.3}$$

The finite-dimensional reduction can now be repeated like in Section 3 with slight changes.

Lemma 7.1. *There are constants* $\varepsilon_6, C_{15} > 0$, *and smooth functions* $(w(\varepsilon,\theta), u(\varepsilon,\theta), \mu(\varepsilon,\theta)) : (-\varepsilon_6, \varepsilon_6) \times \mathbb{R} \to C_0(\mathbb{R},\mathbb{R}) \times \tilde{E}_4 \times \mathbb{R}$ *such that*

(i) $S_{\varepsilon,\theta}(w_\varepsilon(\theta), u_\varepsilon(\theta)) = (\eta_\varepsilon(\theta) G_\varepsilon(\dot{x}_\theta), 0)$;
(ii) $(w_\varepsilon(\theta), \dot{x}_\theta)_{L^2} = 0$;
(iii) $\|w_\varepsilon(\theta)\|_\infty, \|u_\varepsilon(\theta)\|_{E_4} \leq C_{15}\varepsilon$ *for all* $0 < \varepsilon \leq \varepsilon_6$ *and* $\theta \in \mathbb{R}$.

Proof. The proof can be performed like Lemma 3.1 observing that

$$\|L_\varepsilon((w+u)^2 \sin x_\theta)\|_{E_4} \leq C\|w+u\|_{E_4}$$

and that $M_j(t,w,u) = -\sin(x_\theta)\int_0^1 (w(t) + u(t,z))^2 \cos(\pi jz)\,dz + Q_j(t,w,u)$, where $Q(t,w,u) = \sum_{j\geq 0} Q_j(t,w,u)\cos(j\pi z)$ satisfies

$$\|Q(t,u+w)\|_{C_N^4} \leq C'\|w+u\|_{C_N^4}^3.$$

This last expression can be obtained using the Taylor expansion

$$\sin(x_\theta + h) = \sin x_\theta + (\cos x_\theta)h - \frac{(\sin x_\theta)h^2}{2} - \left(\int_0^1 \cos(x_\theta + sh)\frac{(1-s)^2}{2}\right)h^3. \qquad \square$$

We get

Lemma 7.2. *Let* $0 < \varepsilon \leq \varepsilon_6$. *If* $\eta_\varepsilon(\bar\theta) = 0$ *then* $S_{\varepsilon,\bar\theta}(w_\varepsilon(\bar\theta), u_\varepsilon(\bar\theta)) = 0$ *and then* $x_{\bar\theta} + w_\varepsilon(\bar\theta) + y_\varepsilon(\bar\theta)$ *is an heteroclinic solution of system* $(\mathscr{SG}_\varepsilon)$ *connecting* $-\pi$ *to* $+\pi$.

Finally, we can show that the Melnikov function is

$$\mathscr{M}(\theta) = \int_{\mathbb{R}} (-p_0(t,x_\theta(t),0) + \delta\dot x_\theta(t))\dot x_\theta(t)\,dt = -\int_{\mathbb{R}} p_0(t,x_\theta(t),0)\dot x_\theta(t)\,dt + \delta A.$$

Moreover, $\mathscr{M}(\theta) = \Gamma'(\theta) + \delta A$, where Γ is the Poincaré–Melnikov primitive defined by

$$\Gamma(\theta) = \int_{\mathbb{R}} W(t,x_\theta(t))\,dt \tag{7.4}$$

with $-p_0(t,x,0) = (d/dx)W(t,x)$.

The existence of zeroes of the Melnikov functions implies the existence of heteroclinic solutions yielding the following theorem

Theorem 7.1. *Assume* (P3). *If* $\mathscr{M}$ *has a topologically simple zero in* $(\bar\theta - R, \bar\theta + R)$ *for some* $\bar\theta \in \mathbb{R}$ *then for* ε *small enough system* $(\mathscr{SG}_\varepsilon)$ *has an heteroclinic solution* ψ_ε *near* $x_0(\cdot - \tilde\theta)$ *with* $\tilde\theta \in (\bar\theta - R, \bar\theta + R)$.

The previous arguments developed for $x_\theta = x_\theta^+$ can be developed also for x_θ^- gaining the same results. Moreover, it is also possible, arguing as in [5], to glue heteroclinic orbits $x_{\theta_i}^+$ and $x_{\theta_j}^-$ in order to find orbits turning between the equilibria $\pm\pi$ leading to the existence of a chaotic dynamics.

Remark 7.1. With assumtion (P3) we have required that the perturbation $P(t,\psi)$ preserved the equilibria $\pm\pi$. Repeating the same arguments of Section 6 we can prove also for $(\mathscr{SG}_\varepsilon)$, in the case of a purely time-dependent perturbation $P(t)$ under some nonresonance hypotheses, the existence of heteroclinic orbits joining small non-constant trajectories, covering in this way the result by Holmes [9].

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